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A semi-classical analogue of the relation between the chiral and the gluon QCD condensates

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ABSTRACT

We present a semi-classical analogue of the relation between chiral and gluon QCD condensates, in which a condensation of a massless scalar field is provided by a classical field, instead of a quantum Yang–Mills field. For the classical field, we choose a gravitational field of a spherically symmetric object. The size of such an object becomes then a classical-physics analogue of the QCD vacuum correlation length, whereas the squared curvature of the gravitational field plays the role of the gluon condensate. Finally, by iterating the emerging contribution to the trace of the energy–momentum tensor, we prove that the energy density of the object remains of the same order of magnitude, i.e. no instability of the system occurs.

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It is nowadays commonly accepted that the two main non-perturbative phenomena in QCD, confinement and chiral symmetry breaking, are not independent of each other. A clear indication in favor of this statement stems from the following relation that holds between the order parameters of these two phenomena, which are the chiral and the chromo-electric gluon condensates $\langle\bar\psi\psi\rangle$ and $\langle(gE_i^a)^2\rangle$ [1]:

$$\langle\bar\psi\psi\rangle \propto -\lambda\big\langle\big(gE_i^a\big)^2\big\rangle. \tag{1}$$

In this expression, λ is the so-called chromo-electric vacuum correlation length, while the concrete value of the order-$\mathcal{O}(1)$ non-universal proportionality coefficient is unimportant. Clearly, this relation is specific for a stochastic vacuum, such as that of QCD, while it cannot hold in a vacuum characterized by some constant chromo-electric field E_i^a. That is, such a "classical" vacuum cannot support either of the two phenomena. Nevertheless, it is legitimate to pose a question of whether other classical fields can lead to the condensation of quantum fields so as to yield a formula for the corresponding condensates similar to Eq. (1). In this Letter, we show that a simple example of such a classical field is provided by the gravitational field of a spherically symmetric object of a constant energy density. For our illustrative purposes, it suffices to calculate the condensate of a scalar field, which is minimally coupled to this gravitational field. In the vanishing-mass limit of this scalar field, the role of the vacuum correlation length is played by the size of the object. We will obtain a formula for the scalar-field condensate through the second order in the curvature $\mathcal{R}$ and the Ricci tensor $\mathcal{R}_{\mu\nu}$. Within the present analogy, these two quantities play the role of classical counterparts of E_i^a. In order to derive such a formula, we use the known closed-form expression [2] for the one-loop effective action of a scalar field in the gravitational background.

Hence, let us consider a real-valued massive scalar field $\phi(x)$ interacting with the gravitational field $g_{\mu\nu}(x)$. The corresponding Euclidean action has the form

$$S=\frac{1}{2}\int d^4x\sqrt{g}\phi\big(-\Box+m^2\big)\phi, \quad \text{where}$$

$$\Box\phi=\frac{1}{\sqrt{g}}\partial_\mu\big(\sqrt{g}g^{\mu\nu}\partial_\nu\phi\big), \quad \text{and} \quad g\equiv\det g_{\mu\nu}.$$

Integrating over the field ϕ, one obtains the following effective action:

$$\Gamma[g_{\mu\nu}]\equiv-\ln\int\mathcal{D}\phi e^{-S}=\frac{1}{2}\ln\det\big(-\Box+m^2\big).$$

In Ref. [2], a closed-form expression for $\Gamma[g_{\mu\nu}]$ has been obtained through the second order in the curvature:

$$\Gamma[g_{\mu\nu}]=\frac{1}{2}\int_0^\infty\frac{ds}{s}\int\frac{d^4x\sqrt{g}}{(4\pi s)^2}$$

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$$\times\left\{1+s\left(\frac{\mathcal{R}}{6}-m^2\right)+\frac{(ms)^2}{2}\left(m^2-\frac{\mathcal{R}}{6}\right)\right.$$
$$\left.+s^2\left[m^2 f_2\mathcal{R}+\mathcal{R}f_3\mathcal{R}+\mathcal{R}_{\mu\nu}f_4\mathcal{R}^{\mu\nu}\right]\right\}. \tag{2}$$

In this formula,

$$f_2\equiv-\frac{f}{6}+\frac{f-1}{2\xi},\qquad f_3\equiv\frac{f}{32}-\frac{f-1}{8\xi}-\frac{f_4}{8},$$
$$f_4\equiv\frac{f-1-\frac{1}{6}\xi}{\xi^2}, \tag{3}$$

where $f\equiv\int_0^1 du\,\mathrm{e}^{u(1-u)\xi}$, and $\xi\equiv s\Box$. In what follows, we will be interested in the small-m limit, where the s-series in Eq. (2) can be recovered from the full factor $\mathrm{e}^{-m^2 s}$. To order $\mathcal{O}(s^2)$, this approximation yields

$$\Gamma[g_{\mu\nu}]\simeq\frac{1}{2}\int_0^\infty\frac{ds}{s}\mathrm{e}^{-m^2 s}\int\frac{d^4x\sqrt{g}}{(4\pi s)^2}\left\{1+\frac{s\mathcal{R}}{6}\right.$$
$$\left.+s^2\left[\frac{m^2\mathcal{R}}{12}+m^2 f_2\mathcal{R}+\mathcal{R}f_3\mathcal{R}+\mathcal{R}_{\mu\nu}f_4\mathcal{R}^{\mu\nu}\right]\right\}. \tag{4}$$

As we will see below, the $\mathcal{O}(s^2)$-term yields the leading contribution to the scalar-field condensate in the physically interesting case where the size of a spherical object that produces the gravitational field is sufficiently small, being therefore analogous to the chromo-electric vacuum correlation length in QCD. The relation

$$\langle\phi^2\rangle=-2\frac{\partial\Gamma[g_{\mu\nu}]/\partial m^2}{\int d^4x\sqrt{g}} \tag{5}$$

allows us further to obtain, from the effective action (4), the scalar-field condensate $\langle\phi^2\rangle$. A conceptually similar technique, based on the one-loop QCD effective action, has been used in Ref. [3] in order to relate the quark and gluon condensates away from the heavy-quark limit. Nevertheless, as it has already been mentioned, QCD differs from the present model due to the fact that the non-perturbative Yang–Mills fields, being quantum in nature, should be averaged over, whereas the gravitational field in our case is classical.[2] Furthermore, unlike Ref. [3], we work here in the small-m limit.

In order to calculate the effective action (4), we use the following integral representations of the formfactors f_2, f_3, and f_4:

$$f_2=-\frac{1}{6}\int_0^1 du\,\mathrm{e}^{u(1-u)\xi}+\frac{1}{2}\int_0^1 du\,u(1-u)\int_0^1 d\alpha\,\mathrm{e}^{\alpha u(1-u)\xi}, \tag{6}$$

$$f_3=\frac{1}{32}\left[\int_0^1 du\,\mathrm{e}^{u(1-u)\xi}-4\int_0^1 du\,u(1-u)\int_0^1 d\alpha\right.$$
$$\left.\times\left[1+u(1-u)(1-\alpha)\right]\mathrm{e}^{\alpha u(1-u)\xi}\right], \tag{7}$$

$$f_4=\int_0^1 du\left[u(1-u)\right]^2\int_0^1 d\alpha(1-\alpha)\mathrm{e}^{\alpha u(1-u)\xi}. \tag{8}$$

These representations are similar to those which were used in Ref. [3]. Some details of their derivation from Eqs. (3) can be found in Appendix A. Differentiating then the effective action (4) with respect to m^2, we have

$$-\frac{\partial\Gamma[g_{\mu\nu}]}{\partial m^2}\simeq\frac{1}{2}\int_0^\infty ds\,\mathrm{e}^{-m^2 s}\int\frac{d^4x\sqrt{g}}{(4\pi s)^2}\left\{1+\frac{s\mathcal{R}}{12}-sf_2\mathcal{R}\right.$$
$$\left.+s^2\left[\frac{m^2\mathcal{R}}{12}+m^2 f_2\mathcal{R}+\mathcal{R}f_3\mathcal{R}+\mathcal{R}_{\mu\nu}f_4\mathcal{R}^{\mu\nu}\right]\right\}. \tag{9}$$

Furthermore, as it is known [4], the ultraviolet-divergent terms $1+\frac{s\mathcal{R}}{12}-sf_2\mathcal{R}$ in Eq. (9) can be renormalized by adding to $\Gamma[g_{\mu\nu}]$ the sum of the bare Einstein–Hilbert and the cosmological-constant actions. As a result, the gravitational constant gets renormalized independently of the cosmological constant. In the context of the present Letter, the net effect of this renormalization procedure amounts to using the standard value of the gravitational constant, $G=6.7\cdot10^{-39}$ GeV^{-2}, and subtracting from Eq. (9) the three aforementioned ultraviolet-divergent terms. Moreover, since the terms $s^2[\frac{m^2\mathcal{R}}{12}+m^2 f_2\mathcal{R}]$ in Eq. (9) are suppressed in the small-m limit of interest, they will be henceforth disregarded. Thus, we arrive at the following intermediate expression:

$$-\frac{\partial\Gamma[g_{\mu\nu}]}{\partial m^2}\simeq\frac{1}{2(4\pi)^2}\int_0^\infty ds\,\mathrm{e}^{-m^2 s}\int d^4x\sqrt{g}\big(\mathcal{R}f_3\mathcal{R}$$
$$+\mathcal{R}_{\mu\nu}f_4\mathcal{R}^{\mu\nu}\big). \tag{10}$$

As outlined above, we consider now the metric corresponding to the inner part of a spherically symmetric object of radius R, which is filled with the matter of a constant energy density ε. In this case, the metric itself, as well as the associated scalar curvature $\mathcal{R}$ and the Ricci tensor $\mathcal{R}_{\mu\nu}$, are the functions of the radial coordinate $\mathbf{r}$. In Appendix B, we summarize some known facts about this metric, and calculate the corresponding scalar curvature. As follows from this calculation, the adopted approximation, where the terms $\frac{m^2\mathcal{R}}{12}+m^2 f_2\mathcal{R}$ in Eq. (9) are disregarded in comparison with the terms $\mathcal{R}f_3\mathcal{R}+\mathcal{R}_{\mu\nu}f_4\mathcal{R}^{\mu\nu}$, is translated in the inequality

$$m\ll\sqrt{\varepsilon G}. \tag{11}$$

Furthermore, since we are interested in the terms $\mathcal{O}(\mathcal{R}^2)$, $\mathcal{O}(\mathcal{R}_{\mu\nu}\mathcal{R}^{\mu\nu})$, and not in higher-curvature terms, the formfactors f_3 and f_4 in Eq. (10) can be taken at $\xi=0$. Indeed, in the case of $\xi\neq0$, one has, for instance in the term $f_3\mathcal{R}$, the following heat-kernel integral:

$$\mathrm{e}^{\tau\Delta}\mathcal{R}|_{\mathrm{x}}=\frac{1}{(4\pi\tau)^{3/2}}\int d^3y\,\mathrm{e}^{-\frac{y^2}{4\tau}}\mathcal{R}(\mathbf{x}-\mathbf{y}), \tag{12}$$

where $\tau=u(1-u)$ or $\tau=\alpha u(1-u)$. Once expanded in $\mathbf{y}$, the scalar curvature $\mathcal{R}(\mathbf{x}-\mathbf{y})$ yields $\mathcal{R}(\mathbf{x})+y^2\mathcal{O}(\mathcal{R}^2(\mathbf{x}))$, since the term linear in $\mathbf{y}$ vanishes upon the integration. This observation shows that we should restrict ourselves to the leading term of the heat-kernel expansion, which is indeed equivalent to setting $\xi=0$. Then the formfactors f_3 and f_4 become just numbers, namely

$$f_3|_{\xi=0}=\frac{1}{120},\qquad f_4|_{\xi=0}=\frac{1}{60}. \tag{13}$$

Moreover, since we consider the propagation of the field ϕ only to radial distances $r\le R$, the upper limit of the proper-time integration should be restricted by some value $s_{\max}=\gamma R^2$, where γ is a dimensionless constant of the order of unity. The s-integration

[2] Had QCD been equivalent to the present model, one could make a conjecture that the average over the Yang–Mills fields can play the role of a specific classical background, which, unlike the standard classical chromo-electric field, is capable to produce chiral condensate.

yields then a factor of $\frac{1}{m^2}(1-\mathrm{e}^{-\gamma m^2R^2})$, which ensures that $\langle\phi^2\rangle\to 0$ in the limit of $R\to 0$.

We can now proceed to the calculation of $\langle\phi^2\rangle$. By making use of Eqs. (5) and (10), as well as the values (13), it can be represented in the form

$$\langle\phi^2\rangle=\frac{1-\mathrm{e}^{-\gamma m^2R^2}}{120(4\pi)^2m^2}\cdot\frac{I_2+2I_3}{I_1}, \tag{14}$$

where we have denoted by I_1, I_2, and I_3 the following Euclidean integrals:

$$I_1\equiv\int\limits_{r<R}d^3x\sqrt{g},\qquad I_2\equiv\int\limits_{r<R}d^3x\sqrt{g}\mathcal{R}^2,$$
$$I_3\equiv\int\limits_{r<R}d^3x\sqrt{g}\mathcal{R}_{\mu\nu}\mathcal{R}^{\mu\nu}. \tag{15}$$

Here, we have explicitly used the fact that, in the spherically-symmetric case at issue, the metric $g_{\mu\nu}$, as well as the corresponding $\mathcal{R}_{\mu\nu}$ and $\mathcal{R}$, are all functions of the radial coordinate **r**, and not of time. For this reason, the time of observation factors out of I_1, I_2, and I_3, being further canceled out between the numerator and the denominator of Eq. (14). That enabled us to substitute d^4x by d^3x all over in the subsequent equations (15).[3]

Using the inequalities (11) and (29), we observe that $mR\ll 1$, which, in turn, yields

$$\langle\phi^2\rangle\simeq\frac{\gamma R^2}{120(4\pi)^2}\cdot\frac{I_2+2I_3}{I_1}.$$

Referring the reader for the details to Appendix C, we give here the result of the calculation of the integrals I_1, I_2, and I_3 in the limit of $y\ll 1$, where $y=cR^2$, and the parameter c is defined by Eq. (27). It reads

$$\langle\phi^2\rangle\simeq\frac{\gamma R^2}{120(4\pi)^2}\cdot 27c^2=\frac{\gamma}{10}(\varepsilon GR)^2. \tag{16}$$

Thus, in the small-m limit (11), we have obtained an m-independent expression for the condensate $\langle\phi^2\rangle$. Its parametric dependence resembles that of the chiral quark condensate in QCD, given by Eq. (1). Indeed, according to Eqs. (27) and (36), $\mathcal{R}\propto\varepsilon G$, so that $\langle\phi^2\rangle\propto\mathcal{R}^2$, in agreement with the initial equation (10). Thus, $\mathcal{R}^2$ is analogous to $\langle(gE_i^a)^2\rangle$ in what concerns the role played by the gravitational and the chromo-electric fields as catalysts of, respectively, the ϕ-field and the quark-field condensation. We also notice that, in the limit of $y\ll 1$ at issue, the size R of the object is analogous to the correlation length λ, i.e. to the shortest distance scale of the non-perturbative chromo-electric vacuum. The parameter y resembles the quantity [5] $\lambda^2\langle(gE_i^a)^2\rangle^{1/2}\simeq 0.1$, whose numerical smallness, provided by the smallness of λ, ensures the convergence of the cumulant expansion in QCD. In our model, the smallness of y ensures that the leading contribution to $\langle\phi^2\rangle$ is given by Eq. (10).

Let us now consider the back-reaction produced by the $\langle\phi^2\rangle$-condensate on the energy density ε. To this end, we notice that the contribution of the condensate (16) to the v.e.v. of the trace of the energy–momentum tensor is $\langle T^\mu{}_\mu\rangle=m^2\langle\phi^2\rangle\propto(m\varepsilon GR)^2$. This contribution should be added to the classical expression for the trace of the energy–momentum tensor, $T^\mu{}_\mu=\varepsilon-3p$. As one can see, for the case of sufficiently small y's at issue, $3p$ entering $T^\mu{}_\mu$ can be disregarded w.r.t. ε. Indeed, the pressure p, given by Eq. (26), can be approximated by its average value

$$\bar p\equiv\frac{1}{R}\int\limits_0^R dr\,p=\frac{\varepsilon}{2}\int\limits_0^1\frac{dt}{\sqrt{t}}\frac{\sqrt{1-yt}-\sqrt{1-y}}{3\sqrt{1-y}-\sqrt{1-yt}},$$

where we have changed the integration variable r to $t\equiv\frac{z}{y}$. Calculating the latter integral, we obtain

$$\frac{3\bar p}{\varepsilon}=-3+6\times\frac{3\sqrt{1-y}\left[\operatorname{arccot}\sqrt{\frac{8-9y}{y}}+\arctan\left(3\sqrt{\frac{y}{8-9y}}\right)\right]-\sqrt{8-9y}\arcsin\sqrt{y}}{\sqrt{9y+1+\frac{1}{y-1}}}.$$

This is a monotonic function of y, which increases linearly at $y\ll 1$, remaining smaller than 1 for $y\lesssim 0.65$. Thus, for $y\ll 1$, one can indeed approximate $T^\mu{}_\mu$ by ε. Accordingly, it is ε that receives through the quantum correction $\langle T^\mu{}_\mu\rangle$ a contribution $\propto(m\varepsilon GR)^2$.

Recalculating now $\langle\phi^2\rangle$ with the so-corrected ε, and further iterating this procedure, we arrive at the equation

$$\frac{d\varepsilon_n}{dn}\propto(mGR)^2\varepsilon_n^2, \tag{17}$$

where we have denoted by n the cardinal of iterations, and approximately replaced Δn by dn. The solution to Eq. (17), $-\frac{1}{\varepsilon_n}+\frac{1}{\varepsilon}\propto(mGR)^2n$, yields a value of n,

$$n_*\propto\frac{1}{\varepsilon\cdot(mGR)^2}, \tag{18}$$

which looks critical in the sense that $\varepsilon_n\to\infty$ for $n\to n_*$. However, this does not happen, i.e. the energy density does not experience an infinite increase for $n\to n_*$. Indeed, as soon as ε_n starts increasing, the radius of the object also becomes n-dependent, and scales according to Eq. (29) as $R_n\sim\frac{1}{\sqrt{\varepsilon_nG}}$. Therefore, Eq. (17) at $n\to n_*$ takes the form $\frac{d\varepsilon_n}{dn}\propto m^2G\varepsilon_n$, with the solution

$$\varepsilon_n\propto\varepsilon\cdot\mathrm{e}^{m^2Gn}. \tag{19}$$

The mass m of the ϕ-field is bounded from above according to the inequality (11), where ε should not be replaced by ε_n, since ε is in any case smaller than ε_n. Therefore, Eq. (18) yields

$$n_*\gtrsim\frac{1}{\varepsilon^2G^3R^2}\gtrsim\frac{1}{\varepsilon G^2}.$$

Substituting this estimate into Eq. (19), we obtain

$$\varepsilon_n\propto\varepsilon\cdot\mathrm{e}^{\mathrm{const}\cdot\frac{m^2}{\varepsilon G}}\sim\varepsilon,$$

where const ~ 1, and at the last step we have used the inequality (11) once again. Thus, at $n\to n_*$, ε_n remains of the order of ε, i.e. an infinite increase of ε_n does not occur.

In conclusion, we have provided an interesting semi-classical analogue of the relation that holds in QCD between the chiral and the chromo-electric condensates. Within this analogue, the role of the chromo-electric condensate is played by the squared curvature of the classical gravitational field produced by a spherically symmetric object of a constant energy density, while the role of the chromo-electric vacuum correlation length is played by the size of that object, which is considered to be sufficiently small. Finally, by estimating the back-reaction of the so-obtained scalar-field condensate on the energy density ε, recalculating the condensate with such a corrected ε, and iterating this procedure, we have shown that the resulting ε_n remains of the order of the initial ε, i.e. no instability of the system, that could be associated with an infinite increase of its energy density, occurs.

[3] Note that the same fact that all the gravitational quantities in the Tolman–Oppenheimer–Volkoff metric depend entirely on **r** enabled us to use the three-dimensional vectors, and to substitute the d'Alembertian by the Laplacian, already in the earlier equation (12).

Appendix A. Integral representations of the formfactors f_2, f_3, and f_4

In this appendix, we derive integral representations (6)–(8) for the formfactors f_2, f_3, and f_4 from the initial expressions (3). We start with the formfactor f_2. In this case, the corresponding equation (6) stems from the following elementary transformation:

$$\frac{f-1}{\xi}=\int\limits_0^1 du\,u(1-u)\frac{e^{u(1-u)\xi}-1}{u(1-u)\xi}=\int\limits_0^1 du\,u(1-u)\int\limits_0^1 d\alpha e^{\alpha u(1-u)\xi}.$$

We proceed now to the formfactor $f_4=\frac{f-1}{\xi^2}-\frac{1}{6\xi}$. The first term in this expression can be identically rewritten as

$$\frac{f-1}{\xi^2}=\int\limits_0^1 du\big[u(1-u)\big]^2\frac{e^{u(1-u)\xi}-1}{[u(1-u)\xi]^2}.$$

Next, we make use of the formula

$$\frac{e^A-1}{A^2}=\frac{e^A}{A}-\int\limits_0^1 d\alpha\,\alpha e^{\alpha A},$$

in which we further represent $\frac{e^A}{A}$ as

$$\frac{e^A}{A}=\int\limits_0^1 d\alpha\, e^{\alpha A}+\frac{1}{A},$$

and obtain

$$\frac{e^A-1}{A^2}=\frac{1}{A}+\int\limits_0^1 d\alpha(1-\alpha)e^{\alpha A}.$$

Substituting $A=u(1-u)\xi$, we have

$$\frac{f-1}{\xi^2}=\int\limits_0^1 du\,u(1-u)\left[\frac{1}{\xi}+u(1-u)\int\limits_0^1 d\alpha(1-\alpha)e^{\alpha u(1-u)\xi}\right].$$

Noticing that $\int_0^1 du\,u(1-u)=\frac{1}{6}$, we arrive at Eq. (8).

Finally, in order to obtain Eq. (7) from the second equation (3), we substitute into that equation $\frac{f-1}{8\xi}=\frac{1}{4}(f_2+\frac{f}{6})$ from the first equation (3). This yields $f_3=-\frac{1}{96}(f+24f_2+12f_4)$. Using for f_2 and f_4 in this expression the corresponding representations (6) and (8), we obtain Eq. (7).

Appendix B. Scalar curvature in the Tolman–Oppenheimer–Volkoff metric

In this appendix, we summarize, for completeness, some known facts [6] about the gravitational metric $g_{\mu\nu}$ in the interior of a spherically-symmetric object filled with the matter of a constant energy density ε, and calculate the corresponding scalar curvature. The energy–momentum tensor $T_{\mu\nu}$ characterizing the matter is supposed to be of a perfect-fluid type: $T_{\mu\nu}=(p+\varepsilon)u_\mu u_\nu-pg_{\mu\nu}$. Here $u_\mu(x)$ is the four-velocity of the fluid, so that $g^{\mu\nu}u_\mu u_\nu=1$. In the local rest frame of the fluid, where $u_\mu=(\sqrt{g_{00}},\mathbf{0})$, the energy–momentum tensor has the following diagonal form:

$$T^\mu{}_\nu=(p+\varepsilon)u^\mu u_\nu-p\delta^\mu{}_\nu=\mathrm{diag}(\varepsilon,-p,-p,-p),$$

where p is the pressure.

We start our consideration with the general case of a non-constant ε. Due to the spherical symmetry of the object at issue, the space-dependence of ε and p is reduced to their dependence on the spatial distance to the center of the object. Therefore, it is natural to place this center to the origin, and introduce the three-dimensional spherical coordinates (r,θ,ϕ), in which $g_{22}=-r^2$ and $g_{33}=-r^2\sin^2\theta$. The remaining metric components can be sought in the form $g_{00}=e^{a(r)}$ and $g_{11}=-e^{b(r)}$, where $a(r)$ and $b(r)$ are some unknown functions. In these coordinates, the non-vanishing components of the Ricci tensor and the scalar curvature read

$$\mathcal{R}^0{}_0=e^{-b}\left(\frac{a''}{2}+\frac{a'}{r}+\frac{a'^2}{4}-\frac{a'b'}{4}\right),$$
$$\mathcal{R}^1{}_1=e^{-b}\left(\frac{a''}{2}-\frac{b'}{r}+\frac{a'^2}{4}-\frac{a'b'}{4}\right),$$
$$\mathcal{R}^2{}_2=\mathcal{R}^3{}_3=e^{-b}\left(\frac{a'-b'}{2r}+\frac{1}{r^2}\right)-\frac{1}{r^2},$$
$$\mathcal{R}=e^{-b}\left(a''+\frac{2(a'-b')}{r}+\frac{a'^2-a'b'}{2}+\frac{2}{r^2}\right)-\frac{2}{r^2},\tag{20}$$

where the prime denotes the derivative with respect to r. The function $b(r)$ can be determined from the Einstein equation $\mathcal{R}^0{}_0-\frac{1}{2}\mathcal{R}=8\pi GT^0{}_0$. This function reads

$$b(r)=-\ln\left(1-\frac{2G\mathcal{M}}{r}\right),\tag{21}$$

where $\mathcal{M}(r)=4\pi\int_0^r dr'r'^2\varepsilon(r')$ is the energy contained inside a sphere of radius r.

The function $a(r)=\ln g_{00}(r)$ can be found by using the covariant conservation of the energy–momentum tensor, $\nabla_\mu T^{\mu\nu}=0$, and assuming the so-called hydrostatic-equilibrium condition, which implies the x_0-independence of p, ε, and u_μ. This yields the following expression for the metric component $g_{00}(r)$:

$$g_{00}(r)=g_{00}(R)\cdot\exp\left[2\int\limits_r^R dr'\,\frac{dp/dr'}{p+\varepsilon}\right],\tag{22}$$

where

$$g_{00}(R)=1-\frac{r_g}{R}.\tag{23}$$

In the last formula, $r_g\equiv 2GM$ is the Schwarzschild radius, with $M=4\pi\int_0^R drr^2\varepsilon(r)$ being the full energy of the object. Once combined with the Einstein equation $\mathcal{R}^1{}_1-\frac{1}{2}\mathcal{R}=8\pi GT^1{}_1$, Eq. (22) leads to the following differential equation [6]:

$$-\frac{dp}{dr}=\frac{G\varepsilon\mathcal{M}}{r^2}\left(1-\frac{2G\mathcal{M}}{r}\right)^{-1}\left(1+\frac{p}{\varepsilon}\right)\left(1+\frac{4\pi r^3p}{\mathcal{M}}\right).\tag{24}$$

Together with the equation $\frac{d\mathcal{M}}{dr}=4\pi r^2\varepsilon$ and the equation of state, $p=p(\varepsilon)$, Eq. (24) forms a set of three equations for the three unknown functions, namely p, ε, and $\mathcal{M}$. Substituting now Eq. (24) into Eq. (22), one obtains the metric component $g_{00}(r)$ in terms of the functions $p(r)$ and $\mathcal{M}(r)$:

$$g_{00}(r)=g_{00}(R)\times\exp\left[-2G\int\limits_r^R\frac{dr'}{r'^2}\left(1-\frac{2G\mathcal{M}}{r'}\right)^{-1}\left(\mathcal{M}+4\pi r'^3p\right)\right].\tag{25}$$

We proceed now to the case of $\varepsilon = \text{const}$ of interest, where $\mathcal{M}(r) = \frac{4\pi}{3}\varepsilon r^3$. Given the boundary condition $p(R) = 0$, Eq. (24) can be integrated analytically to yield

$$p(r) = \varepsilon \cdot \frac{\sqrt{1-z} - \sqrt{1-cR^2}}{3\sqrt{1-cR^2} - \sqrt{1-z}}, \tag{26}$$

where we have introduced the notations

$$c \equiv \frac{8\pi}{3}\varepsilon G \tag{27}$$

and

$$z \equiv cr^2. \tag{28}$$

For the denominator in Eq. (26) not to vanish for all $r < R$, one imposes the condition (cf. Ref. [6]) $3\sqrt{1-cR^2} - 1 > 0$, which defines an upper limit for the radius of the object:

$$R \le \frac{1}{\sqrt{3\pi\varepsilon G}}. \tag{29}$$

In terms of the variable z, this condition means that

$$z \le cR^2 \le \frac{8}{9}.$$

Noticing that $cR^2 = \frac{r_g}{R}$, where $r_g = \frac{8\pi}{3}\varepsilon G R^3$ is the Schwarzschild radius in the constant-ε case, we obtain from Eqs. (25) and (23):

$$g_{00}(r) = \left(1 - cR^2\right) \times \exp\left[-\int\limits_z^{cR^2} \frac{dz'}{\sqrt{1-z'}(3\sqrt{1-cR^2} - \sqrt{1-z'})}\right]. \tag{30}$$

The integral in the exponential of Eq. (30) can be calculated analytically, which leads to the following expression:

$$g_{00}(r) = \frac{1}{4}\left(3\sqrt{1-cR^2} - \sqrt{1-z}\right)^2. \tag{31}$$

We proceed now to the calculation of the scalar curvature $\mathcal{R}$. According to Eq. (20), it is defined through the first and the second derivatives of the function $a(r) = \ln g_{00}(r)$. The first derivative follows from Eq. (30) directly,

$$a'(r) = \frac{2cr}{\sqrt{1-z}\,(3\sqrt{1-cR^2} - \sqrt{1-z})}, \tag{32}$$

while the second derivative can be obtained through a straightforward calculation, and reads

$$a''(r) = \frac{2c[3\sqrt{1-cR^2} - (1+z)\sqrt{1-z}\,]}{(1-z)^{3/2}[3\sqrt{1-cR^2} - \sqrt{1-z}\,]^2}. \tag{33}$$

The function $b(r)$, given by Eq. (21), takes in the constant-ε case the form

$$b(r) = -\ln(1-z), \tag{34}$$

so that its derivative is obvious:

$$b'(r) = \frac{2cr}{1-z}. \tag{35}$$

Substituting Eqs. (32)–(35) into Eq. (20), we obtain for the scalar curvature the following result:

$$\mathcal{R} = 2c\left\{\frac{21\sqrt{(1-z)(1-cR^2)} + 18cR^2 + 5z - 23}{[3\sqrt{1-cR^2} - \sqrt{1-z}\,]^2} - 1\right\}. \tag{36}$$

Appendix C. Calculation of the integrals I_1, I_2, and I_3

In this appendix, we give details of the calculation of the integrals (15). To this end, we notice that, in the Euclidean space, $g_{11} = \frac{1}{1-z}$, where the parameter z is defined through Eqs. (27) and (28). Now, using for the metric component g_{00} its expression (31), and denoting $y \equiv cR^2$, we find

$$\begin{aligned} I_1 &= \frac{\pi}{c^{3/2}} \int\limits_0^y dz\,\sqrt{z}\left(3\sqrt{\frac{1-y}{1-z}} - 1\right) \\ &= \frac{\pi}{c^{3/2}}\left[\frac{1}{3}\sqrt{y}(7y-9) + 3\sqrt{1-y}\arcsin\sqrt{y}\right]. \end{aligned} \tag{37}$$

In the limit of $y \ll 1$, one obtains $I_1 = \frac{\pi}{c^{3/2}}[\frac{4y^{3/2}}{3} + \mathcal{O}(y^{5/2})]$. Similarly, by virtue of Eq. (36), we have for the integral I_2:

$$I_2 = 36\pi\sqrt{c}\int\limits_0^y dz\sqrt{\frac{z}{1-z}}\frac{[9\sqrt{(1-y)(1-z)} + 9y + 2z - 11]^2}{(3\sqrt{1-y} - \sqrt{1-z})^3}.$$

In the limit of $y \ll 1$, this integral reads $I_2 = 36\pi\sqrt{c}[\frac{y^{3/2}}{3} + \mathcal{O}(y^{5/2})]$. To calculate the integral I_3, we will make use of the formula

$$\begin{aligned} \mathcal{R}_{\mu\nu}\mathcal{R}^{\mu\nu} &= \left(\mathcal{R}^0{}_0\right)^2 + \left(\mathcal{R}^1{}_1\right)^2 + 2\left(\mathcal{R}^2{}_2\right)^2 \\ &= \frac{e^{-2b}}{16}\cdot\left\{\left[2a'' + a'\left(a' - b' + \frac{4}{r}\right)\right]^2 \right. \\ &\quad + \frac{8}{r^4}\left(2 - 2e^b + a'r - b'r\right)^2 \\ &\quad \left. + \frac{1}{r^2}\left[\left(2a'' + a'^2\right)r - b'\left(4 + a'r\right)\right]^2\right\}, \end{aligned} \tag{38}$$

which follows from Eqs. (20) for the components of the Ricci tensor. Upon the substitution of Eqs. (32)–(35) into Eq. (38), we can simplify the integrand in I_3 to arrive at the following expression:

$$\begin{aligned} I_3 = 36\pi\sqrt{c}\int\limits_0^y dz\,&\frac{\sqrt{z}}{(1-z)(8-9y+z)} \\ &\times\left[z(z+4) + 9\sqrt{(1-y)(1-z)}\right. \\ &\left. + 3y\sqrt{1-z}(2\sqrt{1-z} - 3\sqrt{1-y}) - 5\right]. \end{aligned}$$

Carrying out the integration, we obtain

$$\begin{aligned} I_3 = 24\pi\sqrt{c}\Bigg[&\sqrt{y}(9-10y) + 27(1-y)^{3/2}\arcsin\sqrt{y} - 9(1-y) \\ &\times\sqrt{8-9y}\left(\arctan\sqrt{\frac{y}{8-9y}} + \arctan\left(3\sqrt{\frac{y}{8-9y}}\right)\right)\Bigg]. \end{aligned}$$

In the limit of $y \ll 1$, this expression takes the form $I_3 = 36\pi\sqrt{c}[\frac{y^{3/2}}{3} + \mathcal{O}(y^{5/2})]$, i.e. $I_3 = I_2$ in this limit. Substituting finally I_1, I_2, and I_3 into Eq. (14), we obtain for the condensate $\langle\phi^2\rangle$ expression (16) from the main text.

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